

第九章
重积分
同步练习

9.3

1. 求锥面 $z = \sqrt{x^2 + y^2}$ 被柱面 $z^2 = 2x$ 所割下部分的曲面面积

解
$$\begin{cases} z = \sqrt{x^2 + y^2} \\ z^2 = 2x \end{cases}$$

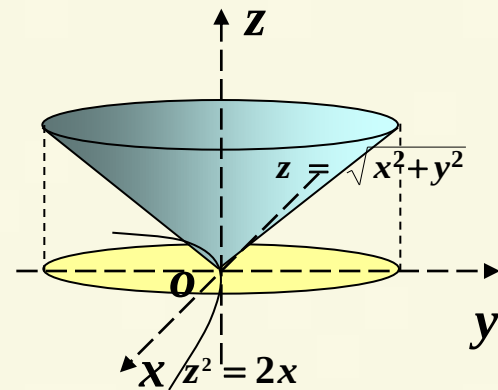
得投影柱面: $x^2 - 2x + y^2 = 0$ D_{xoy}

$$z_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad z_y = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{2}$$

$$S = \iint_{D_{xoy}} \sqrt{1 + z_x^2 + z_y^2} d\sigma = \iint_{D_{xoy}} \sqrt{2} d\sigma = \sqrt{2} \iint_{D_{xoy}} dx dy$$

$$= \sqrt{2} S_{D_{xoy}} = \sqrt{2} \pi$$



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2. 求曲面 $x = yz$ 被柱面 $y^2 + z^2 = 1$ 所割下部分的曲面面积

解

$$\begin{cases} x = yz \\ y^2 + z^2 = 1 \end{cases}$$

得投影柱面: $y^2 + z^2 = 1$ D_{yoz}

$$x_y = z, \quad x_z = y \quad \sqrt{1 + x_y^2 + x_z^2} = \sqrt{1 + z^2 + y^2}$$

$$S = \iint_{D_{yoz}} \sqrt{1 + x_y^2 + x_z^2} d\sigma = \iint_{D_{yoz}} \sqrt{1 + z^2 + y^2} d\sigma$$

$$= \iint_{D_{yoz}} \sqrt{1 + r^2} \cdot r dr d\theta = \int_0^{2\pi} d\theta \int_0^1 (\sqrt{1 + r^2}) r dr$$

$$= \int_0^{2\pi} \left[\frac{1}{3} (1 + r^2)^{\frac{3}{2}} \right]_0^1 d\theta = \int_0^{2\pi} \frac{1}{3} (2\sqrt{2} - 1) d\theta = \frac{2\pi}{3} (2\sqrt{2} - 1)$$

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3. 设球体 $x^2 + y^2 + z^2 \leq 2z$ 上各点的密度等于该点到坐标原点的距离平方, 求该球体的质量.

解 球面坐标系

$$M = \iiint_{\Omega} \rho(x, y, z) dv = \iiint_{\Omega} (x^2 + y^2 + z^2) dv$$

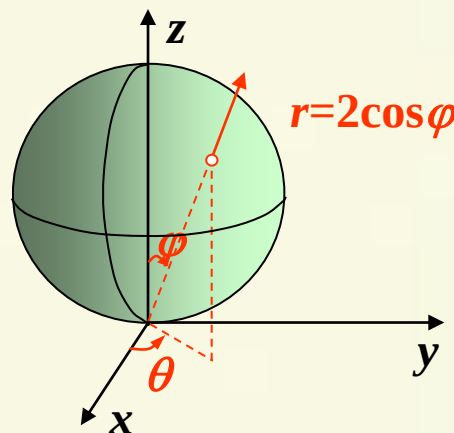
$$= \iiint_{\Omega} r^2 \cdot r^2 \sin \varphi dr d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^4 \sin \varphi dr$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin \varphi \left[\frac{1}{5} r^5 \right]_0^{2\cos\varphi} d\varphi$$

$$= \frac{32}{5} \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin \varphi \cos^5 \varphi d\varphi = \frac{32}{5} \int_0^{2\pi} \left[-\frac{1}{6} \cos^6 \varphi \right]_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{16}{15} \int_0^{2\pi} d\theta = \frac{32}{15} \pi$$

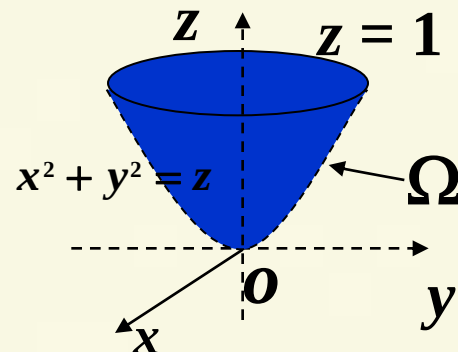


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4. 求占有空间区域 $\Omega = \{(x, y, z) \mid x^2 + y^2 \leq z \leq 1\}$ 的均匀物体形心.

解 设体密度为1, 则

$$\begin{aligned} M &= \iiint_{\Omega} dv = \int_0^1 dz \iint_{x^2+y^2 \leq z} d\sigma \\ &= \int_0^1 \pi z dz = \frac{\pi}{2} [z^2]_0^1 = \frac{\pi}{2} \end{aligned}$$



由对称性可知 $M_{zox} = M_{yoz} = 0$

$$M_{xoy} = \iiint_{\Omega} z dv = \int_0^1 dz \iint_{x^2+y^2 \leq z} z d\sigma = \int_0^1 \pi z^2 dz = \frac{\pi}{3} [z^3]_0^1 = \frac{\pi}{3}$$

从而此立体的形心坐标为

$$\bar{x} = \frac{M_{yoz}}{M} = 0 \quad \bar{y} = \frac{M_{zox}}{M} = 0 \quad \bar{z} = \frac{M_{xoy}}{M} = \frac{2}{3}$$

即 $(0, 0, \frac{2}{3})$

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5. 设有一等腰直角三角形薄片, 腰长为4, 各点处的密度等于该点到直角顶点的距离平方, 求该薄片的形心.

解 $M = \iint_D \rho(x, y) d\sigma = \iint_D (x^2 + y^2) d\sigma$

$$= \int_0^4 dx \int_0^{4-x} (x^2 + y^2) dy$$

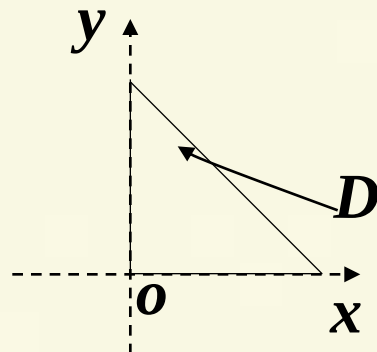
$$= \int_0^4 \left[x^2 y + \frac{1}{3} y^3 \right]_0^{4-x} dx$$

$$= \int_0^4 \left[4x^2 - x^3 + \frac{1}{3} (4-x)^3 \right] dx$$

$$= \left[\frac{4}{3} x^3 - \frac{1}{4} x^4 - \frac{1}{12} (4-x)^4 \right]_0^4 = \frac{2 \cdot 4^3}{3}$$

$$M_x = \iint_D y \rho(x, y) d\sigma = \int_0^4 dy \int_0^{4-y} (x^2 + y^2) y dx$$

$$= \int_0^4 \left[\frac{1}{3} y x^3 + x y^3 \right]_0^{4-y} dy = \int_0^4 \left[\frac{1}{3} y (4-y)^3 + 4y^3 - y^4 \right] dy$$



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5. 设有一等腰直角三角形薄片, 腰长为4, 各点处的密度等于该点到直角顶点的距离平方, 求该薄片的形心.

解 $M = \iint_D \rho(x, y) d\sigma = \frac{2 \cdot 4^3}{3}$

$$M_x = \int_0^4 \left[\frac{1}{3} y(4-y)^3 + 4y^3 - y^4 \right] dy$$

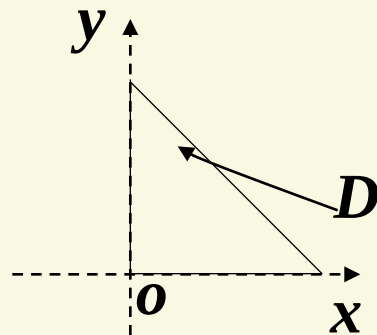
$$= \int_0^4 \left[\frac{1}{3} y(64 - 48y + 12y^2 - y^3) + 4y^3 - y^4 \right] dy$$

$$= \frac{1}{3} \int_0^4 (64y - 48y^2 + 24y^3 - 4y^4) dy$$

$$= \frac{1}{3} \left[32y^2 - 16y^3 + 6y^4 - \frac{4}{5}y^5 \right]_0^4 = \frac{4^5}{15} = M_y = \iint_D x\rho(x, y) d\sigma$$

从而此薄片的形心坐标为

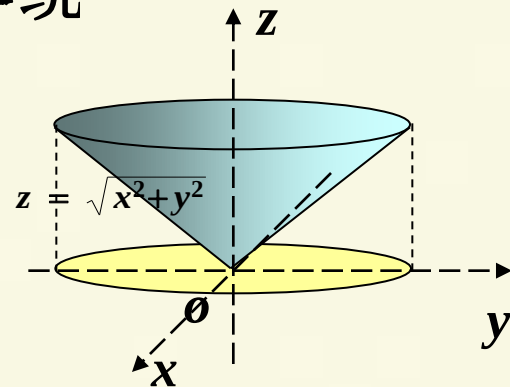
$$\bar{x} = \frac{M_y}{M} = \frac{8}{5} \quad \bar{y} = \frac{M_x}{M} = \frac{8}{5} \quad \text{即} \quad \left(\frac{8}{5}, \frac{8}{5} \right)$$



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6. 求高为 h , 半顶角为 $\frac{\pi}{4}$ 的均匀正圆锥体绕其对称轴旋转的转动惯量.

解 Ω 是 Z -型区域 先二后一



$$I_z = \iiint_{\Omega} (x^2 + y^2) \rho dv$$

$$= \int_0^h dz \iint_{r \leq z} r^2 \cdot r \rho dr d\theta$$

$$= \rho \int_0^h dz \int_0^{2\pi} d\theta \int_0^z r^3 dr = \rho \int_0^h dz \int_0^{2\pi} \left[\frac{1}{4} r^4 \right]_0^z d\theta$$

$$= \frac{\rho}{4} \int_0^h z^4 dz \int_0^{2\pi} d\theta = \frac{\pi}{2} \rho \int_0^h z^5 dz = \frac{\pi}{10} \rho [z^5]_0^h = \frac{\pi}{10} h^5 \rho$$